*ISBN: 978-93-48091-58-1*

# 10. Metric Spaces: A Brief Introduction

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***Abstract:***

*A metric space is a fundamental concept in mathematics that provides a rigorous framework for understanding distance and proximity between points. It generalizes the notion of distance between objects, extending it beyond Euclidean spaces to encompass a wide range of both physical and abstract spaces. This chapter will unveil the foundational concepts of metrics and metric spaces. Some examples along with real-life applications have been provided to understand and appreciate the abstract nature of the topic.*

***Keywords:***

*Real number, Distance, Metric space, Hamming distance*

## 10.1 Introduction:

### 10.1.1 Origin of Metric Spaces:

The notion of distance has always been useful, be it in the convergence of sequences or in limits we have come across so far, but in the limited scope of real analysis. To extend this idea for say, vectors, complex numbers functions, etc, or to get more abstract and define a space that contains all of these objects, two classes fit our requirements - the first is the so-called metric spaces and the second, a more general class of spaces, topological spaces.

In the realm of mathematics, a metric space is a conceptual collection equipped with a distance function, referred to as a metric, which defines a non-negative distance between any pair of its points, while adhering to specific properties. Metric spaces were first introduced by Maurie Frechet in his PhD dissertation on functional analysis, in 1906 (Taskvic, 2005).

The mathematicians of that time were studying various spaces and for each space, its own notion of convergence was introduced. The dire need to simplify things and unify arguments was the motivation for Frechet to axiomatize the notion of distance and show that many of these spaces were instances of metric spaces.

***Why study metric spaces:***

Functional analysis, a theoretical branch of mathematics rooted in classical analysis, emerged approximately eighty years ago. In the present day, its techniques and outcomes hold significance across diverse mathematical disciplines and their practical implementations.

Mathematicians observed that problems from different fields such as linear algebra, linear ordinary and partial differential equations, calculus of variations, approximation theory, and linear integral equations enjoy related features and properties.

The utilization of this fact allowed for an efficient method of unification in addressing these issues, achieved by excluding unnecessary particulars. Therefore, the benefit of adopting such an abstract approach lies in its focus on the crucial elements.

In an abstract approach, one usually starts from a set of elements satisfying certain axioms instead of specifying the nature of elements. Using the axiomatic method, one acquires a mathematical framework, the theory of which is formulated in an abstract manner. The general theorems can then later be applied to various special sets satisfying those axioms.

For instance, within the realm of algebra, this method is utilized in association with fields, rings, and groups.

Moreover, metric spaces hold a foundational position in functional analysis due to their analogous function to the real line $\mathbb{R}$ in calculus. In reality, they extend the concept of $\mathbb{R}$ and have been introduced to establish a foundation for a cohesive approach to significant problems spanning multiple domains of analysis.

### 10.1.2 Metric Spaces:

In this section, first of all, we are going to review the absolute value of real numbers along with straight line distance to extract an idea of ‘distances’ and then generalize this concept to define metric spaces.

***Review of absolute value of real number:***

We now focus on an important function from real analysis, called the absolute function of real numbers. It is defined as $|\,.\,|: \mathbb{R} \to [0, \infty[$ and satisfies the following properties:

i) $|x| = 0$ iff $x = 0, x \in \mathbb{R}$.
ii) $|-x| = |x|, \forall\, x \in \mathbb{R}$.
iii) $|xy| = |x||y|, \forall\, x, y \in \mathbb{R}$.

iv) $|x + y| \leq |x| + |y|, \forall\, x, y \in \mathbb{R}$ .

Physically, $|\,.\,|$ measures the distance of any real number *x* from the real number 0. We utilize this absolute value function to measure the distance between points, especially to study convergent sequences,

Cauchy sequences and continuity of functions. We know that to measure the distance between two reals *x* and *y*, it suffices to measure the length of line segment joining *x* and *y*. But we only know to measure the length by measuring it from 0. So, we shift one of *x* and *y*, say *x* to 0.

Then consequently, $y$ gets shifted to $(y - x)$ and the length of the real no. $(y - x)$ is $|y - x|$ which gives the distance between *x* and *y*.


**Figure 10.1: Measuring the distance between two reals *x* and *y* by the line segment joining them.**


**Figure 10.2: Measuring the distance between two reals *x* and *y* by the line segment after shifting *x* to 0.**

With this notion, we observe the following properties of distance between two reals *x* and $y$.

i) $|x - y| \geq 0$ and $|x - y| = 0$ iff $x = y$; for all $x, y \in \mathbb{R}$.
ii) $|x - y| = |y - x|$; for all $x, y \in \mathbb{R}$.
iii) $|x - y| \leq |x - z| + |z - y|$; for all $x, y, z \in \mathbb{R}$.

### *10.1.3 Notion of Metric Space:*

Basically, a metric space is a non- empty set equipped with a well-defined notion of distance. The term ‘metric’ is derived from the word metor which means ‘measure’.

So, what do we mean by measure? what and how it can be measured? In pursuit of finding solutions, let’s consider the subsequent example:

Suppose a person wishes to travel from Guwahati to New Delhi.


**Figure 10.3: Route from Guwahati to New Delhi.**

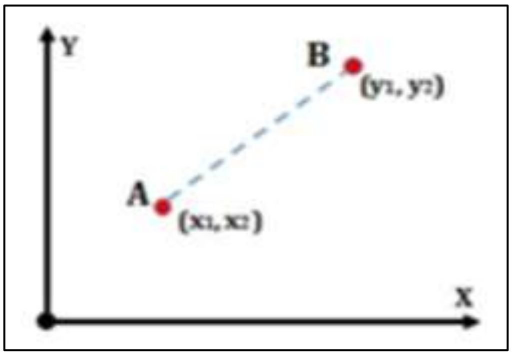
Source: https://www.google.co.in/maps/dir/Guwahati/Delhi

The above Figure 10.3 gives the possible routes from Guwahati to New Delhi. On the basis of the situation, one may choose any of the given possible option. Then his journey can be interpretated in two different ways:

i) The distance in kilometres between Guwahati and New Delhi in terms of navigation.
ii) The distance hours between Guwahati and New Delhi in terms of navigation.

Suppose that the person travels via NH 27, then the time taken is 35 hrs and the distance travelled is 1932 kms. Therefore, in this instance, time and distance symbolize distinct measurement methods.

Now, let us mention the basic characteristic of straight-line distance measured between two points in $\mathbb{R}^2$.



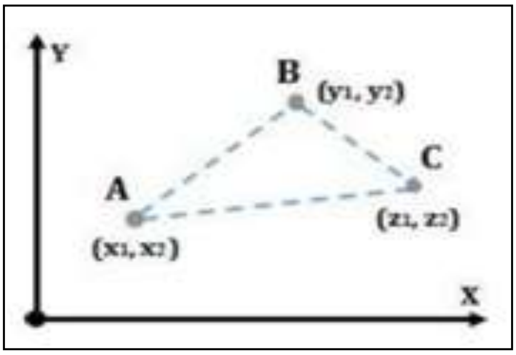




**Figure 10.4: Distance between two points. Figure 10.5: Distance among three points.**

It has been known to us that straight line distance between A$(x_1, y_1)$ and B$(x_2, y_2)$ is $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ . Also,

i) Measurement between two distinct points is always a real positive number.
ii) Two points are the same only when the measurement between them is zero.
iii) Symmetry in nature is reflected in measurement, i.e., distance from A to B is equal to the distance from B to A.
iv) The measurement between two points is either less than or equal to the overall measurement taken when travelling through another point.

Now, how to generalize these notions in an abstract setting so that the properties remain intact? For this we use real valued functions, known as metric.

Definition 10.1 (Deshmukh et al., 2021) (Metric): For any set *X*, let us define a mapping $d: X \times X \to \mathbb{R}$ which satisfies the following conditions:

(M1) For all $a, b \in X$ , $d(a, b) \geq 0$.

(M2) For all $a, b \in X$, $d(a, b) = 0 \; iff a = b$.

(M3) For all $a, b \in X$,$d(a, b) = d(b, a)$.

(M4) For all $a, b, c \in X$, $d(a, c) \leq d(a, b) + d(b, c)$.

Then the function *d* is called a **metric** on *X*, also known as distance function on *X* and the ordered pair $(X, d)$ is called a **Metric space**.

***10.1.4 Examples of Metric Space:***

Example 10.1 (Malik and Arora, 2010) (The Euclidean metric on ℝ) Let us consider a function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ , defined by $d(x, y) = |x - y|$, where ℝ is the set of all real numbers and $x, y \in \mathbb{R}$. Then *d* is a metric on ℝ. This Euclidean metric *d* is also called the usual or standard metric on ℝ. This notion of the distance function is generally used to evaluate limits of functions or sequences of real numbers.

Example 10.2 (Malik and Arora, 2010) (The Euclidean metric on ℂ) Let ℂ be the set of all complex numbers. Let us define a function $d: \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ as $d(z_1 - z_2) = |z_1 - z_2|$ , where $z_1, z_2$ are in ℂ.Then *d* is a metric on ℂ.

This metric is known as the usual metric or Euclidean metric on ℂ.We observe that *d* is an extension to $\mathbb{C} \times \mathbb{C}$ of the usual metric *d* on ℝ.

Example 10.3 (Sharma, 2000) (The Euclidean metric on $\mathbb{R}^2$) Let $\mathbb{R}^2$ be the set of all ordered pairs of real number and let us define the function $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $\mathbb{R}^2$. Then *d* is a metric on $\mathbb{R}^2$.

Example 10.4 (Sharma, 2000) (Taxi cab metric on $\mathbb{R}^2$) Let us define a function $d: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ on the set $\mathbb{R}^2$ of all ordered pairs of real numbers defined by $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$. Then $d$ is a metric on $\mathbb{R}^2$.

The taxicab metric also known as the Manhattan distance (Deshmukh et al., 2021) can be visualised as the distance between the points A and B in the adjoining map (Source: https://www.translatorscafe.com/static/ucvt/img/calc-two-points-distance-10.png ) of a city where the streets are in a grid like form. Finding the Euclidean distance in this case is impractical as any taxi or person can move from A to B only along the streets, rather than moving as the crow flies.


**Figure 10.6: The Manhattan distance and Euclidean distance between two places.**

Example 10.5 (Malik and Arora, 2010) (Discrete metric space) For any non – empty set *X*, the function $d: X \times X \rightarrow \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} 1\,; if\ x \neq y \\ 0\,; if\ x = y \end{cases}$$

is the discrete metric on *X*.

Example 10.6 (Sharma, 2000) (Metric on the set $\mathbb{R}^n$) For the set of all ordered n-tuples of real numbers $\mathbb{R}^n$, the mapping $d: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, defined by $d(x,y) = max\{|x_1 - y_1|, |x_2 - y_2|, \dots\dots, |x_n - y_n|\}$

$$= \max_{1 \leq i \leq n}\{|x_i - y_i|\},$$

where $x = (x_1, x_2, \dots\dots. x_n)$ and $y = (y_1, y_2, \dots\dots, y_n)$ is a metric in $\mathbb{R}^n$.

Example 10.7 (Malik and Arora, 2010) (Usual metric on $\mathbb{R}^n$) For any two points $x = (x_1, x_2, \dots . x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in $\mathbb{R}^n$ , let us define the mapping $d: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$d(x, y) = \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

Then $d$ is a metric and $(\mathbb{R}^n, d)$is a metric space.

Example 10.8 (Simmons, 1963) (The $l_p$space) Let $l_p$ denote the set of all sequences $x = \langle x_n \rangle$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty, p \geq 1$ . For $x = \langle x_n \rangle$ and $y = \langle y_n \rangle$ in $l_p$ , the mapping $\psi: l_p \times l_p \rightarrow \mathbb{R}$

defined by $\psi(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$ is a metric on $l_p$.

**10.2 Application of Metric Spaces:**

***10.2.1 Navigation of Flights (Singh and Aggarwal, 2016):***

We all may have wondered why a specific flight follows a certain route, isn't it? When it comes to long distance flights, the routes are meticulously crafted for optimal efficiency, allowing them to travel from one end of the world (point A) to the other (point B) while covering the least possible distance. In the realm of Euclidean geometry, the shortest path between any two points is a straight line connecting them. However, when dealing with a spherical object like the Earth, the distance between two points is measured along the surface of the sphere. Given the Earth's spherical nature, our travel is restricted to moving along its surface, resembling a circular trajectory. As a result, it becomes essential to devise a method for determining the most direct route between these points.


**Figure 10.7: Location of two places A and B.**

Source: https://qph.cf2.quoracdn.net/main-qimg-ba5311613e0b00313b0e17405f244c68

**Figure 10.8: Flight route from place A to B.**

Source: https://www.aeronavesavenda.com/wp-content/uploads/2018/12/NAT-EGGX-Shanwick-Oceanic-300x161.png )

In the above figures (Fig 10.7 and Fig 10.8), a flight from place A to B seems like appearing to follow an extensive path, but in reality, this distance is the most minimal distance separating two cities. This is due to the fact that on a sphere, distances can only be measured along its exterior surface rather than a direct line cutting through the sphere's interior.

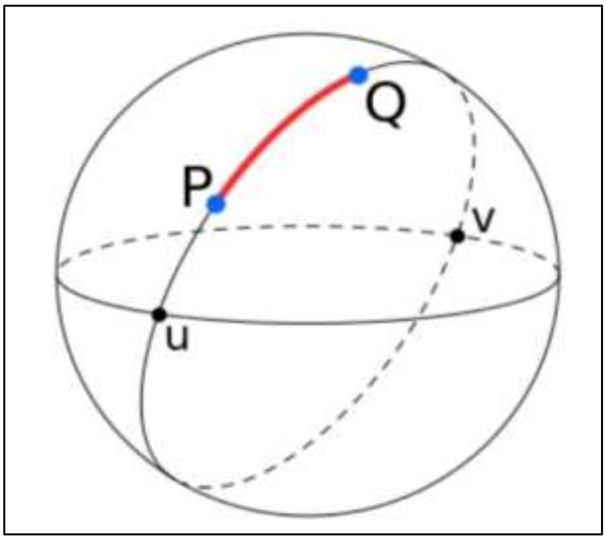



**Figure 10.9: Points in a sphere**

For any pair of points **P** and **Q** on a sphere that are not directly opposite to each other, there passes a unique great circle, i.e., a circular path along Earth's surface whose centre aligns perfectly with the centre of earth. The length of the minor arc between the two points on the great circle is the great circle distance between the points.

Between two points **u** and **v** that are directly opposite each other, there exists an infinite number of great circles and hence the distance separating them equals half the circumference of the great circle.

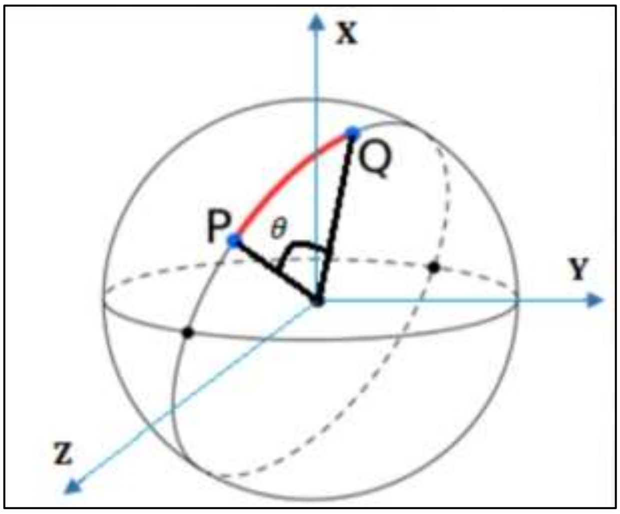



**Figure 10.10: Great circle through PQ**

Most of the long-distance flights use the concept of great circle distance to finalise routes to travel between two locations on the globe. Therefore, on a future occasion when we intend to travel west and the in-flight map displays a northern direction, we are in fact, choosing the most direct (i.e., shortest) route to reach our destination. If S is a sphere and P, Q are in S then let us define $d_s(P,Q)$ as the shortest distance along a great circle connecting points

P and Q, i.e., $d_s(P,Q)$ = length of minor arc PQ

$= r\theta = r\cos^{-1}\left(\frac{p_1q_1+p_2q_2+p_3q_3}{r^2}\right)$, where $r$ = radius of the great circle and $\theta$ = the angle which the arc PQ subtends at the center of the great circle.

It can be verified that $d_s$ is a metric on S. This metric is commonly referred to as the Geodesic metric and it proves highly effective for calculating distances on the Earth's surface.

### *10.2.2 Information Theory:*

During the transmission of data from a sender to a receiver, there exists a significant chance of introducing noise to the transmitted information (i.e., data) in the channel which results in the distortion of the data. In the detection and correction of these errors in the data, Hamming distance is used, named in honour of the American Mathematician Richard Hamming. The Hamming distance is one among the various string metrics employed to quantify the edit distance between two sequences.

When $x$ and $y$ are strings with the equal length, the Hamming distance $D_H(x, y)$ is the number of locations at which they differ. That is, if $x_1x_2 \ldots\ldots x_n$ and $y_1y_2 \ldots\ldots y_n$ are the letters in *x* and *y*, then

$$D_H(x, y) = \{i \in \{1, 2, \ldots, \text{n}\} : x_i \neq y_i\}$$

In information theory, the Hamming distance calculates the smallest count of substitutions needed to transform one string into another, or the smallest possible count of errors that could have converted one string into the other.

For example, given $x = (0, 0, 1, 1, 0, 0, 0, 1, 0)$ and $y = (0, 1, 0, 1, 0, 0, 1, 1, 0)$

It is observed that *x* and *y* vary in the $2^{nd}$, $3^{rd}$ and $7^{th}$ positions and therefore $D_H(x, y) = 3$ .

When considering a constant length n, the Hamming Distance serves as a metric within the collection of words with a specific length n (also known as a Hamming space) as it satisfies the criteria of being non-negative and symmetric. The Hamming Distance of two words is 0 if and only if the two words are identical and it satisfies the triangle inequality as well.

In order to compute the Hamming Distance between the two binary strings, we execute mod 2 addition and subsequently, the cumulative count of '1s' in the resultant string corresponds to the Hamming Distance.

Suppose we have two strings 11011001 and 10011101.

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1 \\ \underline{1\ 0\ 0\ 1\ 1\ 1\ 0\ 1} \\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0 \end{array}$$

Therefore, Hamming distance is 2.

This distance is further utilised in algorithms to correct the accumulated errors.

Another situation that we can think of where Hamming Distance is applicable is in internet search engines. Each one of us has experienced a situation where, upon making an error or typo while using the Google search engine, the engine promptly rectifies the mistake and offers the accurate word as a replacement suggestion. What is the mechanism behind this action of the engine? How does it determine the intended word we meant to type in the query?

The search engine can automatically rectify spelling errors by evaluating the likeness between the strings through the application of the Hamming Distance.

Let us consider these two words: TIME and TAKE.

**The characters of the two strings will be assessed as follows:**

| | | | |
|---|---|---|---|
| T | I | M | E |
| T | A | K | E |

The two strings differ at the 2nd and 3rd positions. So, the Hamming Distance is 2.

The lesser the distance, the more similar are the strings and thus we get our desired result.

### 10.3 Conclusion:

In this chapter, we have discussed about the origin and motivation behind metric spaces and how this concept applied in practical real-life situations. Several metrics on the set of reals $\mathbb{R}$, on $\mathbb{R}^n$, on the set of functions, etc. are included to demonstrate the abstract nature of distance. Finally, we have discussed about some applications of metric spaces in our day-to-day life. The instances where it is being associated with the navigation of flights and in information theory are discussed.

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