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11. Dislocated-Quasi-Modular Metric Space Endowed with Graph and Some Fixed-Point Results Under Quasi-Weak Contraction

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Abstract:

I first define some notion of (α, ψ,a) and (b_n,α,ψ,a)- weak contractive mapping with α-admissible function in dislocated quasi modular metric space endowed with graph. Using this definition to formulate subsequently new results that may generalize and modify some existing results in the literature. Finally, I supply an example of our result for the existence of solution. Presented results would develop, modify and extend several results in the literature.

Keywords:

Comparison function, Dislocated-quasi -modular Metric Space, (α,ψ,a)- weak contraction mapping, directed graph, G-type contractive mapping, α-admissible mapping

11.1 Introduction:

Fixed point theory despite its expanded scope generalization giving the researchers in condensed from not only a wide range but also applicable in a rapidly growing field. Among the different generalizations, Matthew, in 1994 coined the idea of the self-distance $d(x,x)$ is not necessarily zero in partial metric space. Nakano, 1950 coined a new idea of modular in 1950. Christyokov (Chistyakov, 2010a; Chistyakov, 2010b) then developed a new idea of modular metric space having a physical interpretation and established fixed point results in this space. Dislocated quasi-metric is a generalization of the concept of metric spaces. Hitzler (Hitzler, 2001) and Seda (Seda, 2000), introduced dislocated metric space.

Then Zeyada (Zeyada et al., 2006) developed dislocated quasi-metric space and its applications playes an important role in different field like electronic engineering, logic programming etc. and development the literature. Combining these concept Ghosh (Ghosh et al., 2021) deduced dislocated quasi-metric modular space (dqm-metric space). Samet (Samet et al., 2012; Samet et al., 2013) coined the concept of $\alpha-$ admissible mappings.

Later on Karapinar (Karapinar et al., 2014) developed the concept with triangular $\alpha -$ admissible mappings. In this way the study of $\psi -$ contraction mappings are widely researched and extended with the enrichment of the literature. In Erdal, 2013, a new concept on $(\alpha - \psi)$ contraction mapping without Hausdorffness is developed in generalized quasi-metric space.

There are many results on contraction conditions such $(\psi, \phi)-$ contraction, $\phi-$ contraction, $F-$ contraction etc. were introduced and developed by the researchers and prove different interesting results in the area of Fixed point theorem and enrich the literature from last few decades (we refer the reader (Vetro, 2013; Erdal, 2014; Berinde, 2008; Berinde, 2010a; Berinde, 2010b; Karapinar, 2014). It reveals from the literature the successful reporting of fixed point results and their applications in dislocated quasi-metric space as well as dislocated quasi-modular metric space. A vast possibility of this space in the utilization in fixed point theory and new results may bring a generalized way in the field of application such as integral equation, electronic engineering, logical programming, problems in dynamic programming etc.

In this work we investigate existence and uniqueness of a fixed point in quasi modular $(\alpha, \psi, a)-$ weak contraction and quasi modular $(b_n, \alpha, \psi, a)-$ weak contraction mapping in dqm-modular space that generalized several recent results in the literature without the assumption of Hausedorffness.

11.2 Preliminaries:

In order to study fixed point problems on dqm-metric spaces the following basic definitions related to continuity and convergence are needed. We recall these definitions those are useful in the sequel.

Definition 1 (Das et al., 2021; Ozturk and Girgin, 2017)

Let $M \neq \emptyset$ and $\xi, \mu \in (0, \infty)$. A real function $\Theta: (0, \infty) \times M \times M \to [0, \infty)$ of ordered pair of elements of M satisfying the following two conditions for all $p, q, r \in M$

1. $\Theta_\xi(p, q) = \Theta_\xi(q, p) = 0$ for all $\xi > 0 \Rightarrow p = q$

2. $\Theta_{\xi+\mu}(p, q) \leq \Theta_\xi(p, r) + \Theta_\mu(r, q)$ for all $\xi, \mu > 0$

and the pair consisting of two objects M_Θ and Θ_λ is called a dislocated quasi modular metric space (dqm- metric space).

If the first condition in the above definition is replaced by $\Theta_\xi(p, q) = 0$ for all $\xi > 0$ then Θ is called pseudo quasi metric modular and in this case the mapping $\xi \mapsto \Theta_\xi(p, q)$ is decreasing on $(0, \infty)$. Further it is called a regular if the same condition is replaced by $\Theta_\xi(p, q) = 0$ for some $\xi > 0 \Rightarrow p = q$. Θ_ξ is called Non-Archimedian (Vetro et al., 2013) if the second condition is replaced by

$$\Theta_{\max\{\xi,\mu\}}(p, q) \leq \Theta_\xi(p, r) + \Theta_\mu(r, q)$$

A dqm-metric space induced a metric space defined by $\Theta_m = \max\{\Theta_\lambda(p, q), \Theta_\lambda(q, q)\}$

Definition 2 (Ozturk and Girgin, 2017)

Let $(M_\Theta, \Theta_\lambda)$ be a dqm- metric space and $\{p_n\} \subseteq M$. Then the sequence $\{p_n\}$ is a Cauchy sequence if $\lim_{\substack{n\to\infty,\\ m>n}} \Theta_\lambda(p_n, p_m)$ and $\lim_{\substack{n\to\infty,\\ m>n}} \Theta_\lambda(p_m, p_n)$ both exists and finite.

Definition 3 (Ozturk and Girgin, 2017)

Let $(M_\Theta, \Theta_\lambda)$ be a dqm- metric space and $\{p_n\} \subseteq M$. Then the sequence $\{p_n\}$ is a convergent sequence if there exists $s \in M$ such that

$$\lim_{n\to\infty} \Theta_\lambda(p_n, s) = \lim_{n\to\infty} \Theta_\lambda(s, p_n) = \Theta_\lambda(s, s).$$

And we denote by the symbol $\lim_{n\to\infty} p_n = s$.

Definition 4 (Ozturk and Girgin, 2017)

Let $(M_\Theta, \Theta_\lambda)$ be a dqm- metric space and $\{p_n\} \subseteq \Theta_\lambda$. M, Θ_λ is said to be complete if for any Cauchy sequence $\{p_n\} \subseteq M$, there exists $s \in M$ such that

$$\Theta_\lambda(s, s) = \lim_{n\to\infty} \Theta_\lambda(p_n, s)$$

$$= \lim_{n\to\infty} \Theta_\lambda(s, p_n)$$

$$= \lim_{\substack{n\to\infty,\\ m>n}} \Theta_\lambda(p_n, p_m)$$

$$= \lim_{\substack{n\to\infty,\\ m>n}} \Theta_\lambda(p_m, p_n)$$

Definition 5 (Vetro et al., 2012)

Let $\alpha: M \times M \to [0, \infty)$ be a mapping, a mapping $T: M \to M$ is said to be $\alpha -$ admissible if $\alpha(p, q) \geq 1 \Longrightarrow \alpha(Tp, Tq) \geq 1$ for all $p, q \in M$.

Definition 6 (Ozturk and Girgin, 2017)

Let $\alpha: M \times M \to \mathbb{R}^+$ be a mapping, a mapping $T: M \to M$ is said to be sequentially $\alpha -$ admissible if there exists a sub-sequence $\{p_{n_k}\}$ of the sequence $\{p_n\} \subseteq M$ with $\lim_{n\to\infty} p_n = n$ and $\alpha(p_{n+1}, p_n) \geq 1$ and $\alpha(p_n, P_{n+1}) \geq 1 \Rightarrow \alpha(p_{n_k}, p) \geq 1$ and

$\alpha(p, p_{n_k}) \geq 1$ *for all* $p, q \in M, k, n \in \mathbb{N}$.

Definition 7 *Let $\psi: R_0^+ \to R_0^+$ be a non-decreasing mapping and $\sum v_n$ is a convergent series of positive terms such that $\psi^{n+1}(t) \leq \psi^n(t) + v_n$ for all $n \geq m$ in $\mathbb{N}$ and for all $t \in R_0^+ \cup \{0\}$ then $\sum_{n=0}^{\infty} \psi^n(t) < \infty$ for all $t \geq 0$. The function ψ is called compression function.*

We denote all such functions by Ψ_0. Clearly, $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$.

11.3 Main Result:

Definition 8:

Let $(M_\Theta, \Theta_\lambda)$ be a dqm-metric space. A self mapping $T: M_\Theta \to M_\Theta$ is called a quasi modular $(\alpha, \psi, a) -$ weak contractive if

$$\alpha(p,q)\Theta_\lambda(Tp,Tq) \leq \psi(\Theta_\lambda(p,q)) + a\psi(M(p,q)) \qquad (1)$$

where $M(p,q) = min\{\Theta_m(p,Tq), \Theta_m(q,Tp), \Theta_m(p,Tp), \Theta_m(Tq,q)\}, a \geq 0$ and ψ is a comparision function .

Theorem 1:

Let $(M_\Theta, \Theta_\lambda)$ be a complete dqm-metric space and $T: M_\Theta \to M_\Theta$ satisfying the following conditions

1. T is quasi modular $(\alpha, \psi, a) -$ weak contractive
2. T is $\alpha -$ admissible
3. there exists $p_0 \in M_\Theta$ such that $\alpha(p_0, Tp_0) \geq 1$ and $\alpha(Tp_0, p_0) \geq 1$

Then T has a unique fixed point in M_Θ.

Proof. We construct an iterative sequence $\{p_n\}$ of points in M_Θ by setting $p_{n+1} = Tp_n, \quad n \in \mathbb{N} \cup \{0\}$. If for some $n_0 \in \mathbb{N} \cup \{0\}$ such that $p_{n_0+1} = p_{n_0}$ then we have nothing to do. So, let $p_{n+1} \neq p_n$ for all $n \in \mathbb{N} \cup \{0\}$.

We choose $p_0 \in M_\Theta$ such that $\alpha(p_0, Tp_0) \geq 1$ and $\alpha(Tp_0, p_0) \geq 1$. Then by condition (2) one can get,

$$\alpha(p_n, p_{n+1}) \geq 1, \quad n \in \mathbb{N}_0$$

and

$$\alpha(p_{n+1}, p_n) \geq 1, \quad n \in \mathbb{N}_0$$

Now, as $M(p_n, p_{n-1}) = \min\{\Theta_m(p_n, p_n), \Theta_m(p_{n-1}, p_{n+1}), \Theta_m(p_n, p_{n+1}), \Theta_m(p_n, p_{n-1})\}$

So,

$$\Theta_\lambda(p_{n+1}, p_n) = \Theta_\lambda(Tp_n, Tp_{n-1})$$

$$\leq \alpha(p_n, p_{n-1})\Theta_\lambda(Tp_n, Tp_{n-1})$$

$$\leq \psi(\Theta_\lambda(p_n, p_{n-1})) + a\psi(M(p_n, p_{n-1}))$$

$$= \psi(\Theta_\lambda(p_n, p_{n-1}))$$

Because of ψ is non-decreasing, by induction one can have

$$\Theta_\lambda(p_{n+1}, p_n) \leq \psi(\Theta_\lambda(p_n, p_{n-1})) \leq \psi^2(\Theta_\lambda(p_{n-1}, p_{n-2})) \leq \cdots \leq \psi^n(\Theta_\lambda(p_1, p_0))$$

Thus

$$\Theta_\lambda(p_{n+1}, p_n) \leq \psi^n(\Theta_\lambda(p_1, p_0)) \tag{2}$$

By the same analogy one can deduce that

$$\Theta_\lambda(p_n, p_{n+1}) \leq \psi^n(\Theta_\lambda(p_0, p_1)) \tag{3}$$

Letting $n \to \infty$ we have,

$$\lim_{n\to\infty} \Theta_\lambda(p_{n+1}, p_n) = 0 = \lim_{n\to\infty} \Theta_\lambda(p_n, p_{n+1})$$

Next to show that $\{p_n\}$ is a Cauchy sequence.

For that, let $m, n, \in \mathbb{N}$ such that $m > n$. Then

$$\Theta_\lambda(p_n, p_m) = \Theta_{\max\{\lambda,\lambda,\ldots,\lambda\}}(p_n, p_m)$$

$$\leq \Theta_\lambda(p_n, p_{n-1}) + \Theta_\lambda(p_{n-1}, p_{n-2}) + \cdots + \Theta_\lambda(p_{m+1}, p_m)$$

$$\leq \sum_{i=m}^{n-1} \Theta_\lambda(p_{i+1}, p_i)$$

$$\leq \sum_{i=m}^{n-1} \alpha(p_i, p_{i-1})\Theta_\lambda(p_{i+1}, p_i)$$

$$\leq \sum_{i=m}^{n-1} \psi^{i+1}(\Theta_\lambda(p_1, p_0)) + a\psi^{i+1}(M(p_1, p_0))$$

$$\leq \sum_{i=m}^{n-1} \psi^{i+1}(\Theta_\lambda(p_1,p_0))$$

Since ψ is Comparison function so, $\sum_{i=m}^{\infty} \psi^{i+1}(\Theta_\lambda(p_1,p_0))$ is convergent. Thus for a pre-assign positive number ϵ, however small there exists $n \in \mathbb{N}$ such that

$$\sum_{i=m}^{\infty} \psi^{i+1}(\Theta_\lambda(p_1,p_0)) < \epsilon, \quad \forall \quad m \geq N$$

Hence,

$$\Theta_\lambda(p_n,p_m) \leq \sum_{i=m}^{n-1} \psi^{i+1}(\Theta_\lambda(p_1,p_0)) \leq \sum_{i=m}^{\infty} \psi^{i+1}(\Theta_\lambda(p_1,p_0)) < \epsilon$$

In the same manner

$$\Theta_\lambda(p_n,p_m) \leq \sum_{i=n}^{m-1} \psi^{i+1}(\Theta_\lambda(p_0,p_1)) \leq \sum_{i=n}^{\infty} \psi^{i+1}(\Theta_\lambda(p_0,p_1)) < \epsilon$$

Therefore, $\{p_n\}$ is a Cauchy sequence. By completeness of $(M_\Theta, \Theta_\lambda)$, there exists $u \in M_\Theta$ such that

$$\lim_{n\to\infty} \Theta_\lambda(p_n,u) = \lim_{n\to\infty} \Theta_\lambda(u,p_n) = \lim_{n\to\infty} \Theta_\lambda(p_n,p_m) = \lim_{n\to\infty} \Theta_\lambda(p_m,p_n) = 0$$

Now by inequality (3.1) one have,

$$\Theta_\lambda(p_n,Tu) = \Theta_\lambda(Tp_{n-1},Tu)$$

$$\leq \alpha(p_{n-1},u)\Theta_\lambda(p_{n-1},u)$$

$$\leq \psi(\Theta_\lambda(p_{n-1},u)) + a\psi(M(p_{n-1},u))$$

$$< \psi\big(\Theta_\lambda(p_{n-1},u)\big)a\psi(\min\{\Theta_m(p_{n-1},Tu), \Theta_m(u,p_n), \Theta_m(p_{n-1},p_n), \Theta_m(Tu,u)\})$$

Allowing limit as $n \to \infty$, one can have $\lim_{n\to\infty} \Theta_\lambda(p_n,Tu) = 0$. Similarly, $\lim_{n\to\infty} \Theta_\lambda(Tu,p_n) = 0$ This reveals that p_n is convergent to Tu. By uniqueness of limit $Tu = u$.

Uniqueness: If possible suppose that u, v be two fixed points. Then $Tu = u$ and $Tv = v$. By contraction condition (3.1) one can get,

$$\Theta_\lambda(u,v) = \Theta_\lambda(Tu,Tv)$$

$$\leq \alpha(u,v)\Theta_\lambda(u,v)$$

$$\leq \psi(\Theta_\lambda(u,v)) + a\psi(M(u,v))$$

$\leq \psi(\Theta_\lambda(u,v)) +$
$a\psi(\min\{\Theta_m(u,Tv),\Theta_m(v,Tu),\Theta_m(u,Tu),\Theta_m(Tv,v)\})$

$= \psi(\Theta_\lambda(u,v))$

If $\Theta_\lambda(u,v) > 0$ then $\Theta_\lambda(u,v) < \psi(\Theta_\lambda(u,v)) < \Theta_\lambda(u,v)$ which is a contradiction. Therefore $\Theta_\lambda(u,v) \Rightarrow u = v$. This proves the uniqueness.

Remark 1 *If we impose an extra condition that T is continuous. Then also T has an unique fixed point $fu = u$ and it can be obtained by considering the continuity of T as follows:*

$$u = \lim_{n\to\infty} p_{n+1} = \lim_{n\to\infty} fp_n = fu$$

Definition 9 *Let $(M_\Theta, \Theta_\lambda)$ be a dqm-metric space. A self mapping $T: M_\Theta \to M_\Theta$ is called a quasi modular $(b_n, \alpha, \psi, a) -$ weak contractive if*

$$\alpha(p,q)\Theta_\lambda(Tp,Tq) \leq b_n\psi(N(p,q)) + a\psi(M(p,q)) \qquad (4)$$

where $\sum_{n=1}^{\infty} b_n$ with $1 > b_n > 0$ is an infinite series of positive terms, $a \geq 0, \psi$ is a comparision function and

$$N(p,q) = \max\left\{\Theta_\lambda(p,q),\Theta_\lambda(p,Tp),\Theta_\lambda(q,Tq),\frac{\Theta_\lambda(p,Tq)+\Theta_\lambda(q,Tp)}{2}\right\}$$

$$M(p,q) = \min\{\Theta_m(p,Tq),\Theta_m(q,Tp),\Theta_m(p,Tp),\Theta_m(Tq,q)\}$$

Theorem 2 *Let $(M_\Theta, \Theta_\lambda)$ be a complete dqm-metric space and $T: M_\Theta \to M_\Theta$ satisfying the following conditions*

1. T is quasi modular $(b_n, \alpha, \psi, a) -$ weak contraction
2. T is $\alpha -$ admissible
3. there exists $p_0 \in M_\Theta$ such that $\alpha(p_0, Tp_0) \geq 1$ and $\alpha(Tp_0, p_0) \geq 1$

Then T has a unique fixed point in M_Θ.

Proof. We construct an iterative sequence $\{p_n\}$ of points in M_Θ by setting $p_{n+1} = Tp_n, \quad n \in \mathbb{N} \cup \{0\} = N_0$. If for some $r \in \mathbb{N} \cup \{0\} = \mathrm{N}_0$ such that $p_{r+1} = p_r$ then we have nothing to do. So, let $p_{n+1} \neq p_n$ for all $n \in \mathbb{N} \cup \{0\} = \mathrm{N}_0$.

We choose $p_0 \in M_\Theta$ such that $\alpha(p_0, Tp_0) \geq 1$ and $\alpha(Tp_0, p_0) \geq 1$. Then by condition (2) one can get,

$$\alpha(p_n, p_{n+1}) \geq 1, \quad n \in \mathbb{N}_0$$

and

$$\alpha(p_{n+1}, p_n) \geq 1, \quad n \in \mathbb{N}_0$$

Now, as

$$M(p_{n-1}, p_n) = \min\{\Theta_m(p_{n-1}, p_{n+1}), \Theta_m(p_n, p_n), \Theta_m(p_{n-1}, p_n), \Theta_m(p_{n+1}, p_n)\}$$

and $N(p_{n-1}, p_n) = \max\left\{\Theta_\lambda(p_{n-1}, p_n), \Theta_\lambda(p_{n-1}, p_n), \Theta_\lambda(p_n, p_{n+1}), \frac{\Theta_\lambda(p_{n-1}, p_{n+1}) + \Theta_\lambda(p_n, p_n)}{2}\right\}$

$$= \max\left\{\Theta_\lambda(p_{n-1}, p_n), \Theta_\lambda(p_n, p_{n+1}), \frac{\Theta_\lambda(p_{n-1}, p_{n+1})}{2}\right\}$$

$$= \max\left\{\Theta_\lambda(p_{n-1}, p_n), \Theta_\lambda(p_n, p_{n+1}), \frac{\Theta_\lambda(p_{n-1}, p_n) + \Theta_\lambda(p_n, p_{n+1})}{2}\right\}$$

$$= \max\{\Theta_\lambda(p_{n-1}, p_n), \Theta_\lambda(p_n, p_{n+1})\}$$

Therefore,

$$\Theta_\lambda(p_n, p_{n+1}) = \Theta_\lambda(Tp_{n-1}, Tp_n)$$

$$\leq \alpha(p_{n-1}, p_n)\Theta_\lambda(Tp_{n-1}, Tp_n)$$

$$\leq b_n\psi(N(Tp_{n-1}, Tp_n)) + a\psi(M(Tp_{n-1}, Tp_n))$$

$$= b_n\psi(N(Tp_{n-1}, Tp_n))$$

If $N(p_{n-1}, p_n) = \max\{\Theta_\lambda(p_{n-1}, p_n), \Theta_\lambda(p_n, p_{n+1})\} = \Theta_\lambda(p_n, p_{n+1})$ Then

$$\Theta_\lambda(p_n, p_{n+1}) \leq b_n\psi(\Theta_\lambda(p_n, p_{n+1})) \leq b_n\Theta_\lambda(p_n, p_{n+1})$$

This is a contradiction because of the fact $b_n < 1$.

Therefore, $N(p_{n-1}, p_n) = \max\{\Theta_\lambda(p_{n-1}, p_n), \Theta_\lambda(p_n, p_{n+1})\} = \Theta_\lambda(p_{n-1}, p_n)$. Hence

$$\Theta_\lambda(p_n, p_{n+1}) \leq b_n\Theta_\lambda(p_{n-1}, p_n) \tag{5}$$

Again,

$$M(p_n, p_{n-1}) = \min\{\Theta_m(p_n, p_n), \Theta_m(p_{n-1}, p_{n+1}), \Theta_m(p_n, p_{n+1}), \Theta_m(p_n, p_{n-1})\}$$

and

$$N(p_n, p_{n-1}) = \max\left\{\Theta_\lambda(p_n, p_{n-1}), \Theta_\lambda(p_n, p_{n+1}), \Theta_\lambda(p_{n-1}, p_n), \frac{\Theta_\lambda(p_n, p_n) + \Theta_\lambda(p_{n-1}, p_{n+1})}{2}\right\}$$

$$= \max\left\{\Theta_\lambda(p_n, p_{n-1}), \Theta_\lambda(p_n, p_{n+1}), \Theta_\lambda(p_{n-1}, p_n), \frac{\Theta_\lambda(p_{n-1}, p_{n+1})}{2}\right\}$$

$$= \max\left\{\Theta_\lambda(p_n, p_{n-1}), \Theta_\lambda(p_n, p_{n+1}), \Theta_\lambda(p_{n-1}, p_n), \frac{\Theta_\lambda(p_{n-1}, p_n) + \Theta_\lambda(p_n, p_{n+1})}{2}\right\}$$

$$= \max\{\Theta_\lambda(p_n, p_{n-1}), \Theta_\lambda(p_n, p_{n+1}), \Theta_\lambda(p_{n-1}, p_n)\}$$

So, inequality (3.5) implies that,

$$\Theta_\lambda(p_{n+1}, p_n) \leq b_n \max\{\Theta_\lambda(p_n, p_{n-1}), \Theta_\lambda(p_{n-1}, p_n)\} \qquad (6)$$

Again from inequality (3.5) one will have,

$$\Theta_\lambda(p_n, p_{n+1}) \leq b_n \Theta_\lambda(p_{n-1}, p_n)$$

$$\leq b_n \max\{\Theta_\lambda(p_{n-1}, p_n), \Theta_\lambda(p_n, p_{n-1})\} \qquad (7)$$

From (3.6) and (3.7) one have,

$$\max\{\Theta_\lambda(p_n, p_{n+1}), \Theta_\lambda(p_{n+1}, p_n)\} \leq b_n \max\{\Theta_\lambda(p_{n-1}, p_n), \Theta_\lambda(p_n, p_{n-1})\}$$

Implying that,

$$\max\{\Theta_\lambda(p_n, p_{n+1}), \Theta_\lambda(p_{n+1}, p_n)\} \leq b_n \max\{\Theta_\lambda(p_{n-1}, p_n), \Theta_\lambda(p_n, p_{n-1})\}$$

$$\ldots$$

$$\ldots$$

$$\leq b_n^n \max\{\Theta_\lambda(p_0, p_1), \Theta_\lambda(p_1, p_0)\}$$

Hence one have,

$$\Theta_\lambda(p_n, p_{n+1}) \leq b_n K \qquad (8)$$

and

$$\Theta_\lambda(p_{n+1}, p_n) \leq b_n K \qquad (9)$$

Where $K = \max\{\Theta_\lambda(p_n, p_{n-1}), \Theta_\lambda(p_{n-1}, p_n)\}$. Now we show that $\{p_n\}$ is Cauchy sequence. Let $m, n \in \mathbb{N}$ with $m > n$. Then using (3.8) one can get,

$$\Theta_\lambda(p_n, p_m) = \Theta_{\max\{\lambda,\lambda,\ldots,\lambda\}}(p_n, p_m)$$

$$< n(p_n, p_{n-1}) + n(p_{n-1}, p_{n-2}) + \cdots + \Theta_\lambda(p_{m+1}, p_m)$$

$$< \sum_{i=m}^{n-1} \Theta_\lambda(p_{i+1}, p_i)$$

$$\leq \sum_{i=m}^{n-1} \{\alpha(p_i, p_{i-1})\Theta_\lambda(p_{i+1}, p_i)\}$$

$$\leq \sum_{i=m}^{n-1} \left\{b_n^{i+1}K + a\psi^{i+1}(M(p_1, p_0))\right\}$$

$$\leq \sum_{i=m}^{n-1} b_n^{i+1}K$$

$$\leq \sum_{i=m}^{\infty} b_n^{i+1}K$$

Since $\sum_{n=1}^{\infty} b_n^{i+1}$ being an infinite series with $b_n < 1$, is convergent. So, for given a positive pre-assign number ϵ however small a positive number $N \in \mathbb{N}$ can be found such that, $\sum_{n=1}^{\infty} b_n^{i+1} < \frac{\epsilon}{K}$. Therefore for all $m, n, \in \mathbb{N}$ with $n > m \geq N$ for which

$$\Theta_\lambda(p_n, p_m) < \epsilon$$

Similarly, by using (3.7) one can get, $\Theta_\lambda(p_n, p_m) < \epsilon$ for all $m > n \geq N$. Thus $\{p_n\}$ is both left and right Cauchy and consequently a Cauchy sequence.

By completeness of $(M_\Theta, \Theta_\lambda)$, there exists $u \in M_\Theta$ such that

$$\lim_{n\to\infty} \Theta_\lambda(p_n, u) = \lim_{n\to\infty} \Theta_\lambda(u, p_n) = \lim_{n\to\infty} \Theta_\lambda(p_n, p_m) = \lim_{n\to\infty} \Theta_\lambda(p_m, p_n) = 0$$

By inequality (3. 4)

$$\Theta_\lambda(p_n, Tu) = \Theta_\lambda(Tp_{n-1}, Tu)$$

$$\leq \alpha(p_{n-1}, u)\Theta_\lambda(p_{n-1}, u)$$

$$\leq b_n\psi(N(p_{n-1}, u)) + a\psi(M(p_{n-1}, u))$$

$$< b_n\psi\left(\max\left\{\Theta_\lambda(p_{n-1}, u), \Theta_\lambda(p_{n-1}, p_n), \Theta_\lambda(u, Tu), \frac{\Theta_\lambda(p_{n-1}, Tu) + \Theta_\lambda(u, p_n)}{2}\right\}\right)$$

$$+ a\psi(\min\{\Theta_m(p_{n-1}, Tu), \Theta_m(u, p_n), \Theta_m(p_{n-1}, p_n), \Theta_m(Tu, u)\})$$

Allowing limit as $n \to \infty$ we get $\Theta_\lambda(u, Tu) \le b_n \Theta_\lambda(u, Tu)$ which is true only when $\Theta_\lambda(u, Tu) = 0$ because of the reason $b_n < 1$. So that $Tu = u$.

Uniqueness: If possible suppose that u, v be two fixed points. Then $Tu = u$ and $Tv = v$. By contraction condition (3.1) one can get,

$$\Theta_\lambda(u, v) = \Theta_\lambda(Tu, Tv)$$

$$\le \alpha(u, v)\Theta_\lambda(u, v)$$

$$\le b_n \psi(N(u, v)) + a\psi(M(u, v))$$

$$\le b_n \psi\left(\max\left\{\Theta_\lambda(u, v), \Theta_\lambda(u, Tu), \Theta_\lambda(v, Tv), \frac{\Theta_\lambda(u,Tv)+\Theta_\lambda(v,Tu)}{2}\right\}\right)$$

$$+a\psi(\min\{\Theta_m(u, Tv), \Theta_m(v, Tu), \Theta_m(u, Tu), \Theta_m(Tv, v)\})$$

$$= b_n \psi\left(\max\left\{\Theta_\lambda(u, v), \frac{\Theta_\lambda(u,Tv)+\Theta_\lambda(v,Tu)}{2}\right\}\right)$$

Similarly, $\Theta_\lambda(v, u) \le b_n \psi\left(\max\left\{\Theta_\lambda(v, u), \frac{\Theta_\lambda(u,Tv)+\Theta_\lambda(v,Tu)}{2}\right\}\right)$.

Thus

$$\max\{\Theta_\lambda(u, v), \Theta_\lambda(v, u)\} \le b_n \psi\left(\max\left\{\Theta_\lambda(u, v), \Theta_\lambda(v, u), \frac{\Theta_\lambda(u,Tv)+\Theta_\lambda(v,Tu)}{2}\right\}\right)$$

$$= b_n \psi(\max\{\Theta_\lambda(u, v), \Theta_\lambda(v, u)\})$$

$$\le b_n \max\{\Theta_\lambda(u, v), \Theta_\lambda(v, u)\}$$

Because $b < 1$ so, $\Theta_\lambda(u, v) = \Theta_\lambda(v, u) = 0 \Rightarrow u = v$.

Theorem 3 *Let $(M_\Theta, \Theta_\lambda)$ be a complete dqm-metric space and $T: M_\Theta \to M_\Theta$ satisfying the following conditions*

1. T is quasi modular $(b_n, \alpha, \psi, a) -$ weak contraction
2. there exists $p_0 \in M_\Theta$ such that $\alpha(p_0, Tp_0) \ge 1$ and $\alpha(Tp_0, p_0) \ge 1$
3. M_Θ is sequentially $\alpha -$ admissible.

Then T has a unique fixed point in M_Θ.

Proof. As in Theorem 2 the iterative sequence defined therein such that $\lim_{n\to\infty} p_n = u$. Also, $\alpha(p_{n+1}, p_n) \geq 1$ and $\alpha(p_n, p_{n+1}) \geq 1$, $n \in \mathbb{N}_0$. By sequentially $\alpha -$ admissibility of M_Θ there exists a sub-sequence $\{p_{n_k}\}$ of the sequence $\{p_n\} \subseteq M_\Theta$ with $\alpha(p_{n+1}, p_n) \geq 1$ and $\alpha(p_n, p_{n+1}) \geq 1 \Longrightarrow \alpha\left(p_{n_k}, u\right) \geq 1$ and $\alpha(u, p_{n_k}) \geq 1$ for all $p, q \in M_\Theta, k, n \in \mathbb{N}$.

Let for a given arbitrary $\epsilon > 0$ we choose $n_{k_0} \in \mathbb{N}$ such that, the terms $\Theta_\lambda(p_n, u), \Theta_\lambda(u, p_n), \Theta_\lambda(p_n, Tu), \Theta_\lambda(Tu, p_n), \Theta_\lambda(p_m, p_n), \Theta_\lambda(p_n, p_m) < \frac{\epsilon}{2}$ for $m > n \geq n_{k_0}$.

So, for any $k \geq k_0$ by triangle inequality, one will have

$$\Theta_\lambda(u, Tu) \leq \Theta_\lambda(u, p_{n_k+1}) + \Theta_\lambda(p_{n_k+1}, Tu)$$

$$\leq \frac{\epsilon}{2} + \alpha(p_{n_k}, u)\Theta_\lambda(Tp_{n_k}, Tu)$$

$$\leq \frac{\epsilon}{2} + b_n\psi(N(p_{n_k}, u)) + a\psi(M(p_{n_k}, u))$$

where

$$N(p_{n_k}, u) = \max\left\{\Theta_\lambda(p_{n_k}, u), \Theta_\lambda(p_{n_k}, p_{n_k+1}), \Theta_\lambda(u, Tu), \frac{\Theta_\lambda(p_{n_k}, Tu) + \Theta_\lambda(u, p_{n_k+1})}{2}\right\}$$

$$\leq \max\left\{\frac{\epsilon}{2}, \Theta_\lambda(u, Tu), \frac{\Theta_\lambda(p_{n_k}, Tu) + \Theta_\lambda(u, Tu) + \frac{\epsilon}{2}}{2}\right\}$$

$$\leq \frac{\epsilon + \Theta_\lambda(u, Tu)}{2}$$

and

$$M(p_{n_k}, u) = \min\{\Theta_m(p_{n_k}, Tu), \Theta_m(u, p_{n_k+1}), \Theta_m(p_{n_k}, p_{n_k+1}), \Theta_m(Tu, u)\}$$

$$\leq \frac{\epsilon}{2}$$

As ϵ was arbitrary so, $M(p_{n_k}, u) = 0$

$$\Theta_\lambda(u, Tu) \leq \frac{\epsilon}{2} + b_n\psi\left(\frac{\epsilon + \Theta_\lambda(u, Tu)}{2}\right)$$

$$\leq \frac{\epsilon}{2} + b_n\frac{\epsilon + \Theta_\lambda(u, Tu)}{2}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon + \Theta_\lambda(u, Tu)}{2}$$

$$\leq 2\epsilon$$

But $\epsilon > 0$ was arbitrary so, $\Theta_\lambda(u, Tu) = 0$. Hence u is a fixed point of T.

Uniqueness: If possible suppose that u, v be two fixed points. Then $Tu = u$ and $Tv = v$. By contraction condition (3.1) one can get,

$$\Theta_\lambda(u, v) = \Theta_\lambda(Tu, Tv)$$

$$\leq \alpha\Theta_\lambda(Tu, Tv)$$

$$\leq \Theta_\lambda(Tu, p_{n_k+1}) + \Theta_\lambda(p_{n_k+1}, Tv)$$

$$\leq \frac{\epsilon}{2} + \alpha(p_{n_k}, u)\Theta_\lambda(Tp_{n_k}, Tv)$$

$$\leq \frac{\epsilon}{2} + b_n\psi(N(p_{n_k}, v)) + a\psi(M(p_{n_k}, v))$$

Where $N(p_{n_k}, v) \leq \frac{\epsilon + \Theta_\lambda(v, Tv)}{2}$ and $M(p_{n_k}, v) = 0$. So that, $\Theta_\lambda(u, v) = 0$. In the same manner $\Theta_\lambda(v, u) = 0$. Thus $\Theta_\lambda(u, v) = \Theta_\lambda(v, u) = 0 \Rightarrow u = v$.

If we take $a = 0$ then the contraction condition in 4 becomes

$$\alpha(p, q)\Theta_\lambda(Tp, Tq) \leq b_n\psi(N(p, q)) \quad (10)$$

Corollary 1 *Let $(M_\Theta, \Theta_\lambda)$ be a complete dqm-metric space and $T: M_\Theta \to M_\Theta$ satisfying the following conditions*

(i) T is quasi modular $(b_n, \alpha, \psi, a) -$ weak contraction
(ii) T is $\alpha -$ admissible
(iii) there exists $p_0 \in M_\Theta$ such that $\alpha(p_0, Tp_0) \geq 1$ and $\alpha(Tp_0, p_0) \geq 1$

Then T has a unique fixed point in M_Θ.

Example 3.1 *Let $X = M_\Theta = \{0,1,2\}$ and $\Theta_\lambda: X \times X \to [0, \infty[$ defined by*

$\Theta_\lambda(0,0) = 0$, $\Theta_\lambda(0,1) = 1$, $\Theta_\lambda(0,2) = 2$, $\Theta_\lambda(1,0) = 1$, $\Theta_\lambda(1,1) = 1$,

$\Theta_\lambda(1,2) = 2$, $\Theta_\lambda(2,0) = 2$, $\Theta_\lambda(2,1) = 3$, $\Theta_\lambda(2,2) = 4$

Then (X, Θ_λ) is a dqm-metric space. Define $T: X \to X$ by $T(0) = 0, T(1) = 2, T(2) = 0$. Let $\alpha: X \times X \to [0, \infty[$ defined by

$$\alpha(x, y) = \begin{cases} 0, & x \neq 1 \text{or} y \neq 1 \\ 1, & \text{otherwise} \end{cases}$$

and $\psi(t) = \frac{t}{2}, \quad t \geq 0$. Then T is quasi modular $(b_n, \alpha, \psi, a) -$ weak contractive mapping but not $x = y = 1$. Also other requirements of th2 are fulfilled. Hence T has a fixed point. The fixed point of T is $p = 0$ here.

11.4 Fixed Point Theorem Endowed with Graph Theory:

Fixed point theory endow with graph plays prominent role in recent investigations in many different aspects in the literature. In Let M_θ be a dislocated quasi modular metric space and $\Lambda = \{(i, i): i \in M_\theta\}$ diagonals of $M_\theta \times M_\theta$. Let G be a directed graph such that $V(G)$ and $E(G)$ be respectively its vertices set and edges set of the graph G which coincides and $\Lambda \subset E(G)$. The notion and terminology of graph theory one can find in any book of graph theory (see [19, 20]).

If i and j are the vertices of G which connected then there is a path in G from i to j of length $n \in \mathbb{N}$ is a finite sequence of G $\{i_n\}$ of vertices such that $i = i_0, i_1, \ldots, i_n = j$ and $(i_{k-1}, i_k) \in E(G)$ for $i = 1, 2, \ldots, n$. Define G^{-1} by

$$E(G^{-1}) = \{(i, j) \in M_\theta \times M_\theta : (j, i) \in E(G)\}$$

If $\tilde{G}$ denote the undirected graph obtained from G by dropping the direction of the edges of G. Then

$$E(\tilde{G}) = \{E(G) \cup E(G^{-1})\}$$

Let G_i be the component of G with all edges and vertices of G. Denote a relation R in G such that iRj if and only if there is a path from i to j for all $i, j \in V(G)$

Definition 10:

Let $(M_\theta, \omega_\lambda)$ be a dislocated quasi modular metric space and $A, B: M_\theta \to M_\theta$ such that $A(M_\theta) \subseteq B(M_\theta)$. It is said G-type contractive mapping if

1. *A preserves edges of G i.e. for all $(i, j) \in M_\theta \times M_\theta$: $(i, j) \in E(G) \Rightarrow (A(i), A(j)) \in E(G)$ and*

2. *A satisfies contractive condition*

$$\alpha(p, q)\Theta_\lambda(Tp, Tq) \leq \psi(\Theta_\lambda(p, q)) + a\psi(M(p, q)) \qquad (11)$$

where $M(p,q) = min\{\Theta_m(p,Tq), \Theta_m(q,Tp), \Theta_m(p,Tp), \Theta_m(Tq,q)\}, a \geq 0$ *and* ψ *is a comperision function .*

Theorem 4 *Let* M_Ω *be a complete dislocated quasi modular metric space with a graph* G*. Let* $A: M_\theta \to M_\theta$ *and satisfies* G*-type contractive condition e11 , where*

$$M(p,q) = \min\{\Theta_m(p,Tq), \Theta_m(q,Tp), \Theta_m(p,Tp), \Theta_m(Tq,q)\},$$

and ψ is a comperision function. Then A has common unique fixed point.

Proof. Define a sequence $\{i_k\} \in M_\Theta$ by $i_{k+1} = Ai_k$ for all $k \in \mathbb{N}$. Let i_0 be a given point in M_Θ, then $(i_0, Ai_0) = (i_0, i_1) \in E$. Since A preserves the edges of G so,

$$(i_0, i_1) \in G(E) \Rightarrow (Ai_0, Ai_1) \in G(E)$$

Continuing in this way we get $(i_k, i_{k+1}) \in E(G)$. Similarly, we can show that for

$$(i_1, i_0) \in G(E) \Rightarrow (Ai_2, Ai_1) \in G(E), \ldots, (i_k, i_{k-1}) \in G(E) \Rightarrow (Ai_{k+1}, Ai_k) \in G(E)$$

By theorem 1 we can show that $\{i_k\}$ is left-Cauchy sequence as well as right-Cauchy sequence. So by completeness there exists $r \in M_\Theta$ such that $\lim_{n\to\infty} \Theta_\lambda(i_n, r) = \lim_{n\to\infty} \Theta_\lambda(r, i_n) = 0$.

We now show that r is our fixed point. By equation (4.1) we have

$$\Theta_\lambda(Ar, i_{n+1}) = \Theta_\lambda(Ar, Ai_n)$$

$$\leq \alpha(r, i_n)\Theta_\lambda(Ar, Ai_n)$$

$$\leq \psi\big(\Theta_\lambda(Ar, Ai_n)\big) + a\psi\big(M(Ar, Ai_n)\big)$$

On permitting limit as $n \to \infty$ one will obtain that,

$$\Theta_\lambda(Ar, r) = \Theta_\lambda(r, Ar) = 0 \Rightarrow Ar = r$$

Hence r is a fixed point of A.

Theorem 5 *Let* $(M_\Theta, \Theta_\lambda)$ *be a complete dislocated quasi modular metric space with a graph* G*. Let* $T: M_\Theta \to M_\theta$ *be a continuous self mapping satisfying contraction condtions,* T *preserves edges of* G *i.e. for all* $(i,j) \in M_\Theta \times M_\Theta$*:* $(i,j) \in E(G) \Rightarrow (T(i), T(j)) \in E(G)$ *and*

$$\alpha(p,q)\Theta_\lambda(Tp,Tq) \leq a\psi(\Theta_\lambda(p,q)),\ 0 < a < 1$$

Then T has unique fixed point.

Proof. Let $i_0 \in M_\Theta$ such that $i_{k+1} = Ti_k$ for all $k \in N$. Since T preserves the edges of G,

$$(i_0, i_1) \in E(G) \Rightarrow (Ti_0, Ti_1) \in E(G)$$

Continuing in this way we get $(Ti_k, Ti_{k+1}) \in E(G)$. Similarly, we can show that for

$$(i_1, i_0) \in E(G) \Rightarrow (Ti_2, Ti_1) \in E(G), \ldots, (i_k, i_{k-1}) \in E(G) \Rightarrow (Ti_{k+1}, Ti_k) \in E(G)$$

By theorem 1 we can show that $\{i_k\}$ is left-Cauchy sequence as well as right-Cauchy sequence. So there exists $r \in M_\Theta$ such that $\lim_{n\to\infty} \Theta_\lambda(i_n, r) = \lim_{n\to\infty} \Theta_\lambda(r, i_n) = 0$.

$$\lim_{n\to\infty} i_n = Tr = r.$$

Uniqueness:

$$\Theta_\lambda(p, q) = \Theta_\lambda(Tp, Tq)$$

$$\leq \alpha(p, q)\Theta_\lambda(Tp, Tq)$$

$$\leq a\psi(M(p, q))$$

$$\leq \psi(\Theta_\lambda(p, q))$$

Hence $p = q$.

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